The local mapping. The Maximum Modulus Theorem.

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Def. Let $f'(z)$ exists and $f(z)\neq 0$. $\frac{f'}{f}$ is called the logarithmic derivative of f at z.

Heuristics. If $\log f(z)$ is defined, then $(\log f(z))' = \frac{f'(z)}{f(z)}$.

Observe: $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}$. $\frac{(1/f)'}{1/f} = -\frac{f'}{f}$. $\frac{((z-a)^k)'}{(z-a)^k} = \frac{k}{z-a}$ $(k\in\mathbb{Z})$

Let γ be a curve, $f\in\mathcal{A}(\gamma)$, $f(z)\neq 0$ $\forall z\in\gamma$.
$\Gamma := f\circ\gamma$ – piecewise differentiable curve.

Observe: ($z(t)$ – parameterization of γ, $f(z(t))$ – of Γ).

$$n(\Gamma,0) = \frac{1}{2\pi i}\oint_\Gamma \frac{dw}{w} = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))\,z'(t)}{f(z(t))}\,dt =$$

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz$$ – integral of logarithmic derivative.

Let $\gamma\subset B(z_0,r)$ – closed curve. $In_\gamma :=$ Union of bounded components of $\mathbb{C}\setminus\gamma$.

Observe: $\operatorname{Clos} In_\gamma = \gamma\cup In_\gamma$, and
$z\notin \operatorname{Clos} In_\gamma \Rightarrow z$ is in unbounded component of $\mathbb{C}\setminus\gamma \Rightarrow n(\gamma,z)=0$.

Local argument principle

Theorem. Let $f\in\mathcal{M}(B(z_0,r))$, γ – closed curve in $B(z_0,r)$. $f(z)\neq 0,\infty$ on γ.

Then $n(f\circ\gamma,0) = \frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\,dz = \sum_{z\notin\gamma} \operatorname{ord}(f,z)\,n(\gamma,z)$.

Remark. The sum on RHS seems infinite but $z\notin \operatorname{Clos} In_\gamma \Rightarrow n(\gamma,z)=0$,
so the sum is finite: compact $\operatorname{Clos} In_\gamma$ contains only finitely many zeroes and poles.
Also, if z is not zero or pole, $\operatorname{ord}(f,z)=0$.

Take $r'<r$: $B(z_0,r')\supset\gamma$. Then $\overline{B(z_0,r')}\subset B(z_0,r)$, so there are finitely many zeroes or poles inside $B(z_0,r')$

Proof. Let $z_1, z_2, \dots, z_n$ – zeroes and poles of f in ~~In_γ~~ $B(z_0,r')$, with algebraic orders $k_1,\dots,k_n$ respectively.

Observe that the function $g(z) := (z-z_1)^{-k_1}\cdots(z-z_n)^{-k_n} f(z)$ is

1) $g(z)\in\mathcal{A}(B(z_0,r)\setminus\{z_1,\dots,z_n\})$.

2) $\lim_{z\to z_i} g(z) = \prod_{\ell\neq i}(z_i - z_\ell)^{-k_\ell}\cdot \lim_{z\to z_i}\frac{f(z)}{(z-z_i)^{k_i}}$ – exists, $\neq 0,\infty$.

Thus $g(z)\in\mathcal{A}(\cancel{\operatorname{Clos}(In_\gamma)}\ B(z_0,r'))$ $g(z)\neq 0$ $\forall z\in \cancel{\operatorname{Clos}(In_\gamma)}\ B(z_0,r')$

So $\frac{g'(z)}{g(z)}\in\mathcal{A}(\cancel{\operatorname{Clos}(In_\gamma)}\ B(z_0,r'))$

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^n \frac{k_i}{(z-z_i)} + \frac{g'(z)}{g(z)} \Rightarrow$$

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{k_j}{(z-z_j)} + \frac{g'(z)}{g(z)} \Rightarrow$$

$B(z_0, r')$

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} \frac{k_j}{2\pi i}\underbrace{\oint_\gamma \frac{dz}{z-z_j}}_{n(\gamma, z_j)} + \frac{1}{2\pi i}\underbrace{\oint_\gamma \frac{g'(z)}{g(z)}\,dz}_{0}$$

0 by Cauchy. $\frac{g'}{g} \in \mathcal{A}(B(z_0, r'))$

I used Cauchy Theorem for $\frac{g'}{g} \in \mathcal{A}(\mathrm{Clos}(I_{n_\gamma}))$
But we only proved it for $\mathcal{A}(B(z_0, \delta))$
Correct me!

__Corollary__ Let $f \in \mathcal{A}(B(z_0, r))$, $\gamma \subset B(z_0, r)$ - closed curve.
Then $\forall w \in \mathbb{C}$. $n(f \circ \gamma, w) = \sum h_j\, n(\gamma, z_j(w))$, where
$z_j(w)$ are roots of $f(z) = w$ with order h_j. $w \notin f \circ \gamma$.

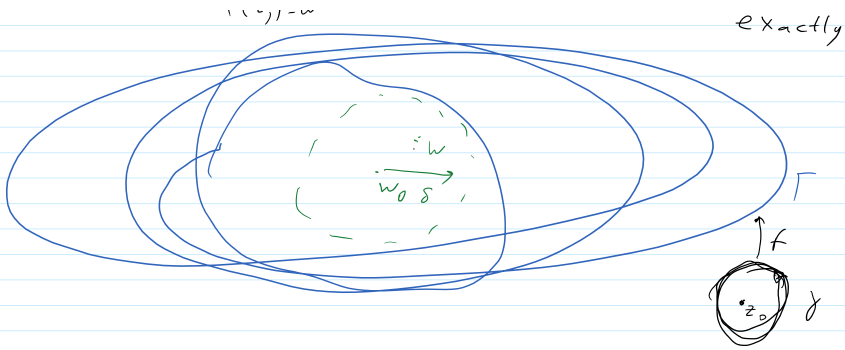
__Proof.__ $n(f \circ \gamma, w) = \frac{1}{2\pi i}\oint_\gamma \frac{f'(\xi)}{f(\xi) - w}\,d\xi$, so we can apply Local Argument Principle to $f(z) - w$ ∎

__Theorem__ (local behavior). Assume that $f \in \mathcal{A}(\Omega)$, $z_0 \in \Omega$, $f(z_0) = w_0$,
and $f(z) - w_0$ has zero of order $n \geq 1$ at z_0 (since $f(z_0) - w_0 = 0$, $n \geq 1$). Then $\exists \varepsilon_0 > 0$: $\varepsilon < \varepsilon_0 \Rightarrow \exists \delta > 0$: $0 < |w - w_0| < \delta \Rightarrow \begin{pmatrix} \exists z_1, z_2, \ldots, z_n \in B(z_0, \varepsilon): \\ \forall j : f(z_j) = w \end{pmatrix}$

__Proof.__ $f' \in \mathcal{A}(\Omega)$, so all zeroes of f' are isolated. All zeroes of $f(z) - w_0$ are also isolated.

Thus $\exists\ \varepsilon_0 : 0 < |z - z_0| < \varepsilon_0 \Rightarrow \begin{matrix} f'(z) \neq 0 \\ f(z) \neq w_0 \end{matrix}$. Take $\varepsilon < \varepsilon_0$.
Take γ to be $\{|z - z_0| = \varepsilon\}$, oriented counterclockwise.
Then $n(\gamma, z) = \begin{cases} 1, & |z - z_0| < \varepsilon \\ 0, & |z - z_0| > \varepsilon \end{cases}$ Let $\Gamma := f \circ \gamma$.
So $n(\Gamma, w_0) = n$ (only z_0 is a zero, of order n).
So $w_0 \notin \Gamma \Rightarrow \delta := \mathrm{dist}(w_0, \Gamma) > 0$.
Take $w \in B(w_0, \delta)$. Then w belongs to the same component of $\mathbb{C} \setminus \Gamma$ as w_0!

So $n(\Gamma, w) = n(\Gamma, w_0) = n$
But $n(\Gamma, w) = \sum_{\substack{|z_j - z_0| < \varepsilon \\ f(z_j) = w}} h_j\, n(\gamma, z_j)$. But $f'(z_j) \neq 0 \Rightarrow h_j = 1$. $n(\gamma, z_j) = 1$. $\Rightarrow$ there are exactly n (z_j)'s ∎



Theorem (analytic maps are open).
Let $\boxed{\begin{array}{l} f\in A(\Omega) \\ f\not\equiv const\end{array}}$ for some region Ω, $U\subset\Omega$ - open $\Rightarrow f(U)$ - open

Restatement: $\forall z_0\in\Omega$, $\forall\ 0<\varepsilon<dist(z_0,\partial\Omega)$
$\exists\delta>0 : (|w-f(z_0)|<\delta \Rightarrow \exists z\in B(z_0,\varepsilon): f(z)=w)$
$\Longleftrightarrow\ f(B(z_0,\varepsilon))\supset B(f(z_0),\delta)$.

Remark. If f is injective on Ω, then $\forall\varepsilon>0\ \exists\delta>0$ $f^{-1}(B(f(z_0),\delta))\subset B(z_0,\varepsilon)$,
so f^{-1} is continuous. $(\forall\varepsilon>0\ \exists\delta>0: f^{-1}(B(w_0,\delta))\subset f(B(f^{-1}(w_0),\varepsilon)))$
$z_0=f^{-1}(w_0)$

Proof. Take $\tilde\varepsilon=\min(\varepsilon,\varepsilon_0)$ from Local Map Theorem.
Then, by the theorem $f(B(z_0,\varepsilon))\supset f(B(z_0,\tilde\varepsilon))\supset B(f(z_0),\delta)$,
for δ from Theorem ∎

Corollary. (Border correspondence).
S - closed, bounded, $\boxed{\begin{array}{l} f\in A(S) \\ f\not\equiv const\end{array}}$. Then $f(\partial S)\supset\partial(f(S))$.
Proof. $w_0\in f(Int(S))\Rightarrow w_0\in Int(f(S))$ (open $\mapsto$ open).
So $w_0\in\partial f(S)\Rightarrow w_0\notin f(Int(S))$. But $f(S)$ - compact, so closed.
So $w_0\in\partial f(S)\Rightarrow \begin{array}{l} w_0\in f(S) \\ w_0\notin f(Int S)\end{array} \Rightarrow w_0\in f(\partial S)$ ∎

Theorem. Let f be a 1-1 analytic function $f:\Omega\to\mathbb{C}$.

Then $f^{-1}: f(\Omega) \to \Omega$ is also analytic.

Proof. If Ω is a region, so is $f(\Omega)$ — it is open (f is open and connected).

$f'(z) \neq 0 \ \forall z \in \Omega$ (by local behavior). By open map theorem, f^{-1} is continuous.

So, by a homework problem, f^{-1} is complex differentiable. ∎

Local coordinate change. Let $f(z) \in \mathcal{A}(\Omega)$, $z_0 \in \Omega$, $f(z) - f(z_0)$ has zero of order n at z_0. Then $\exists \varepsilon > 0$ and a conformal (1-1) $h \in \mathcal{A}(B(z_0, \varepsilon))$: $f(z) - f(z_0) = (h(z))^n$. $h(z_0) = 0$.

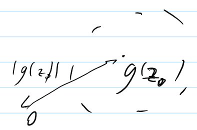
Proof. $f(z) - f(z_0) = (z - z_0)^n g(z)$ for some $g(z) \in \mathcal{A}(\Omega)$. $g(z_0) \neq 0$. Fix $0 < r < \mathrm{dist}(z_0, \partial\Omega)$ such that $|z - z_0| \leq r \Rightarrow f(z) \neq f(z_0)$.

Let $\gamma = \{|z - z_0| = r\}$, oriented counterclockwise.

Then $n(f(\gamma), f(z_0)) = n$.

$\exists \varepsilon > 0$: $|z - z_0| < \varepsilon \Rightarrow$ 1) $z \in \Omega$ 2) $|g(z)| > \frac{|g(z_0)|}{2}$ 3) $|f(z) - f(z_0)| < \mathrm{dist}(f(z_0), f(\gamma))$ ($\Rightarrow n(f(\gamma), f(z)) = n$)

$(r > \varepsilon)$



A branch $\ell(w)$ of $\log w$ is defined in $B\left(g(z_0), \frac{|g(z_0)|}{2}\right)$.

So the function $h(z) := (z - z_0) \exp\left(\frac{\ell(g(z))}{n}\right)$ is well-defined in $B(z_0, \varepsilon)$, analytic in $B(z_0, \varepsilon)$ and satisfies $h(z)^n = (z - z_0)^n \cdot \left(\exp\left(\frac{\ell(g(z))}{n}\right)\right)^n = (z - z_0)^n \exp(\ell(g(z))) = (z - z_0)^n g(z) = f(z) - f(z_0)$.

Note now that for any $z \in B(z_0, \varepsilon)$, since $f(z) - f(z_0) = h(z)^n$,

$n = n(f(\gamma), f(z)) = n \cdot n(h(\gamma), h(z))$, so $n(h(\gamma), h(z)) = 1$, which means that if $z' \neq z$, $|z' - z_0| < \varepsilon \Rightarrow h(z) \neq h(z')$

argument principle

So h is conformal ∎

Theorem (Maximum Principle). Let $f \in \mathcal{A}(\Omega)$, $z_0 \in \Omega$ and $|f|$ reaches a local maximum at z_0 (i.e. $\exists \varepsilon > 0$: $|z - z_0| < \varepsilon, z \in \Omega \Rightarrow |f(z)| \leq |f(z_0)|$).

Then $f \equiv \text{const}$.

Proof. Assume $f \not\equiv \text{const}$. Then $\exists \delta > 0$ $f(\underbrace{B(z_0, \varepsilon)}_{\text{open!}}) \supset B(f(z_0), \delta)$. Take $B(z_0, \varepsilon) \subset \Omega$

So $f(z_0) + \frac{\delta}{2} \frac{f(z_0)}{|f(z_0)|} \in B(f(z_0), \delta) \subset f(B(z_0, \varepsilon))$, so

$\exists z$: $|z - z_0| < \varepsilon$, $z \in \Omega$, $f(z) = f(z_0) + \frac{\delta}{2} \frac{f(z_0)}{|f(z_0)|}$.

$|f(z)| = \left(1 + \frac{\delta}{2|f(z_0)|}\right) |f(z_0)| > |f(z_0)|$ — contradiction!

How to modify it for $f(z_0) = 0$?

How to modify it for $f(z_0)=0$?

Theorem. Let S be closed and bounded.
$f\in C(S)$ – continuous on S.
$f\in A(\mathrm{Int}\,S)$. Then

$$\max_{z\in S}|f(z)| = \max_{z\in\partial S}|f(z)|.$$

If $f\not\equiv$ const, then $\forall z\in \mathrm{Int}\,S$, $|f(z)|<\max_{z\in S}|f(z)|$.

Proof. If $f\equiv$ const – nothing to prove.
If $f\not\equiv$ const, by compactness, $\exists z_0\in S: f(z_0)=\max_{z\in S}|f(z)|$
By Maximum Principle, $z_0\notin \mathrm{Int}\,S$.
So $z_0\in\partial S$, and $\forall z\in \mathrm{Int}\,S$, $|f(z)|<|f(z_0)|$ – again,
by Maximum Principle ∎

Another proof of FTA:

Let $p(z)=a_n z^n+\dots+a_0$, $a_n\neq 0$.
Assume: $\forall z: p(z)\neq 0$.
Consider $f(z):=\frac{1}{p(z)}$ – analytic.
Then $\forall |z|<R$, $|f(z)|\le \max_{|z|=R}|f(z)| = \frac{1}{\min_{|z|=R} p(z)} = m_R$
But as $|z|\to\infty$, $|p(z)| = |z^n|\left|\left(a_n+\dots+\frac{a_0}{z^n}\right)\right|\to \lim_{|z|\to\infty}|z^n|\cdot|a_n|=\infty$,
So as $R\to\infty$, $m_R\to 0$. So $\forall z$ $|f(z)|=0$ – contradiction ∎